Nuts & Bolts of Advanced Imaging

Image Reconstruction – Parallel Imaging

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No conflicts of interest to disclose

Outline

- Noise correlation
- SNR scaled reconstruction
  - Obtaining images in SNR units
- Pseudo Replica Method
  - Determining the SNR (and g-map) for any parallel imaging reconstruction
- Iterative methods
  - Non-Cartesian Parallel Imaging
- Regularization in Iterative Methods

Noise in Parallel Imaging

Idealized Experiment:

$$\mathbf{s} = \mathbf{E}\boldsymbol{\rho}$$

In practice, we are affected by noise

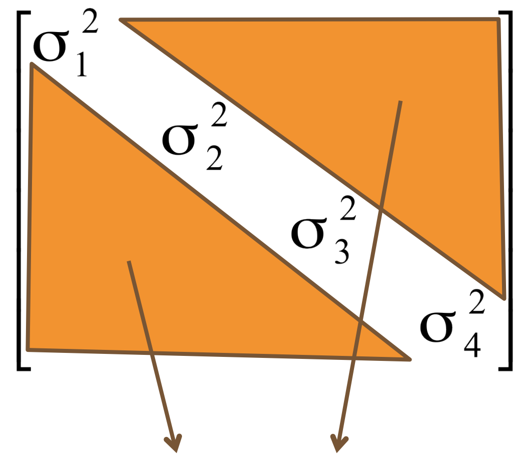
$$\mathbf{s} = \mathbf{E}\boldsymbol{\rho} + \boldsymbol{\eta}$$

We can measure this noise covariance:

```
% Matlab
% eta:[Ncoils, Nsamples]
Psi = (1/(Nsamples-1))*(eta * eta');
```

Noise covariance matrix

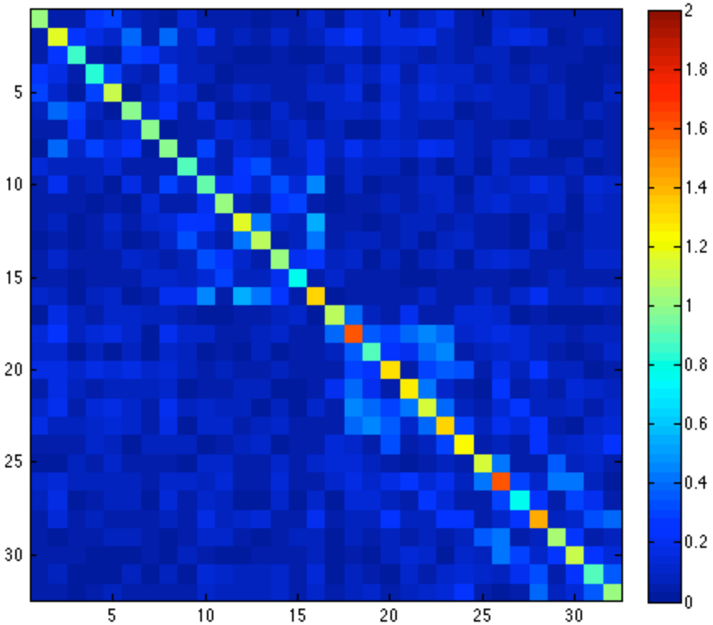
$$\Psi_{\Upsilon,\Upsilon'} = \langle \eta_{\Upsilon}, \eta_{\Upsilon'} \rangle$$



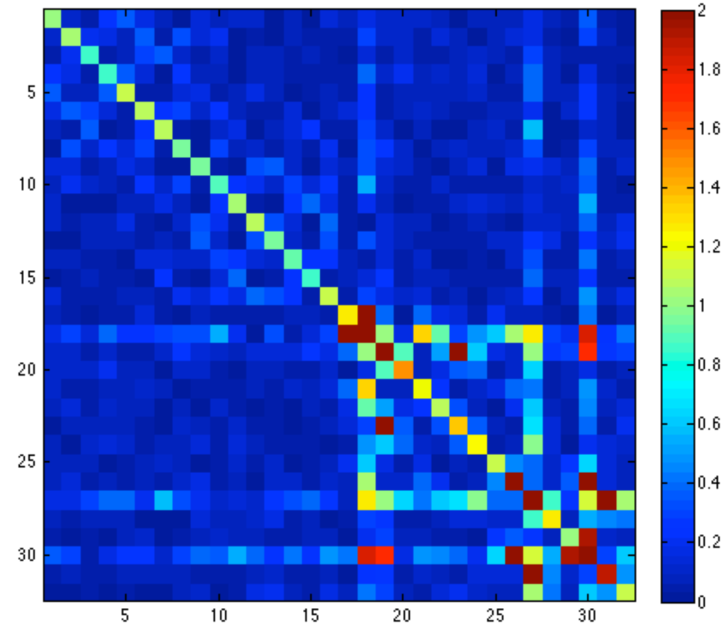
Noise correlation

Psi Examples – 32 Channel Coil

“Normal Coil”



“Broken Coil”



Examination of the noise covariance matrix is an important QA tool. Reveals broken elements, faulty pre-amps, etc.

Noise Pre-Whitening

Solving Linear Equations:

$$\mathbf{Ax} + \boldsymbol{\eta} = \mathbf{b} = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \\ c_5 & c_6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$X_i \,:\, \text{Random value with zero mean } (\mu = 0) \text{ and variance } \sigma_i^2$$

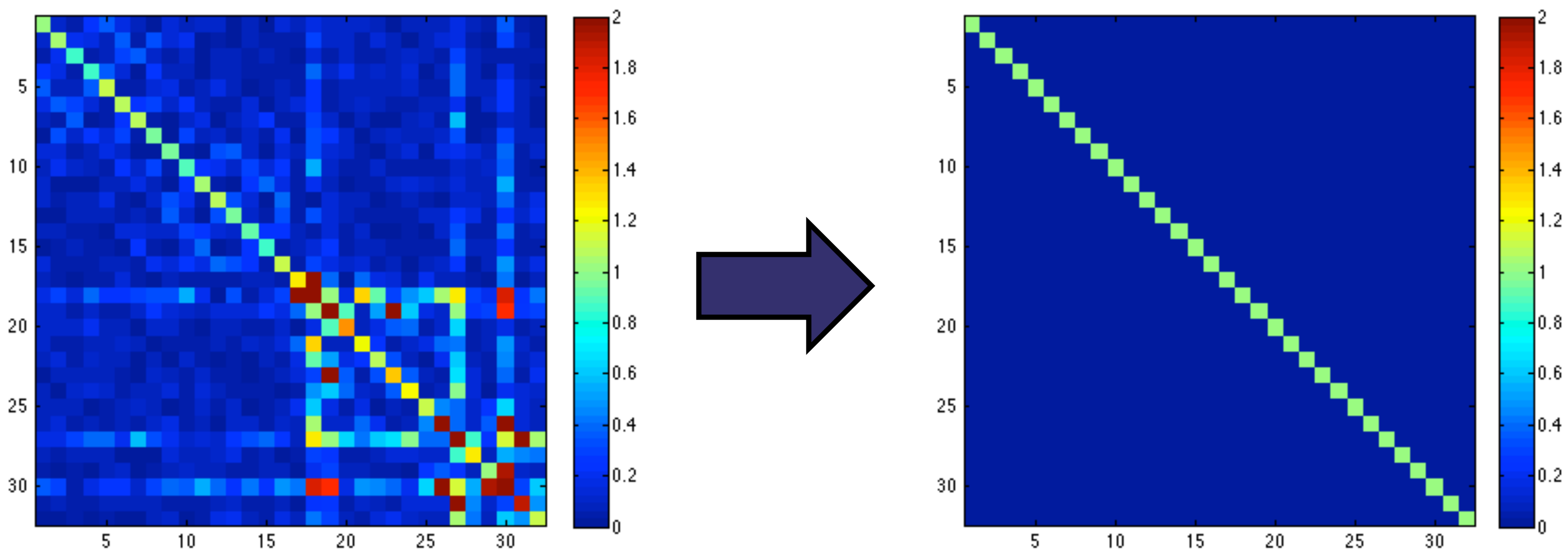
Suppose you know that:

$$\sigma_3^2 = 5\sigma_1^2 = 5\sigma_2^2$$

Put less weight on this equation

Noise Pre-Whitening

We would like to apply an operation such that we have unit variance in all channels:



Noise Pre-Whitening

More generally, we want to weigh the equations with the "inverse square root" of the noise covariance, if

$$\mathbf{\Psi} = \mathbf{L}\mathbf{L}^{\mathrm{H}}$$

We will solve:

$$\mathbf{L}^{-1}\mathbf{A}\mathbf{x} = \mathbf{L}^{-1}\mathbf{b}$$

Or:

$$\mathbf{x} = \left(\mathbf{A}^{\mathrm{H}}\mathbf{\Psi}^{-1}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{H}}\mathbf{\Psi}^{-1}\mathbf{b}$$

In practice, we simply generate "pre-whitened" input data before recon

Noise Pre-Whitening

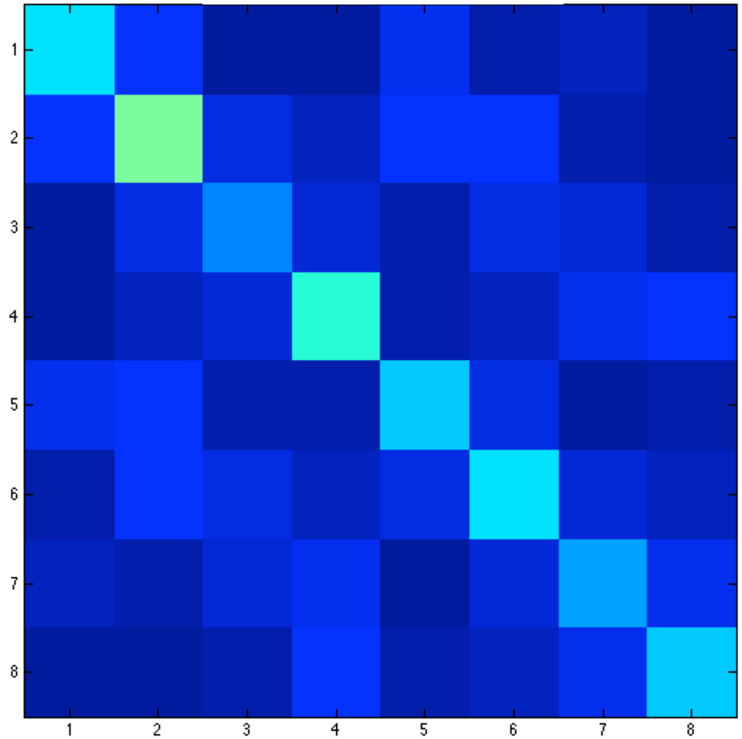
Matlab:

```
%eta [Ncoils,Nsamples]
%psi [Ncoils,Ncoils]
%data [Ncoils,Nsamples]
%csm : Coil sensitivity map

psi = (1/(Nsamples-1))*(eta * eta');

L = chol(psi,'lower');
L_inv = inv(L);

data = L_inv * data;
csm = L_inv * csm;

%Now noise is “white”
%Reshape data and do recon
```

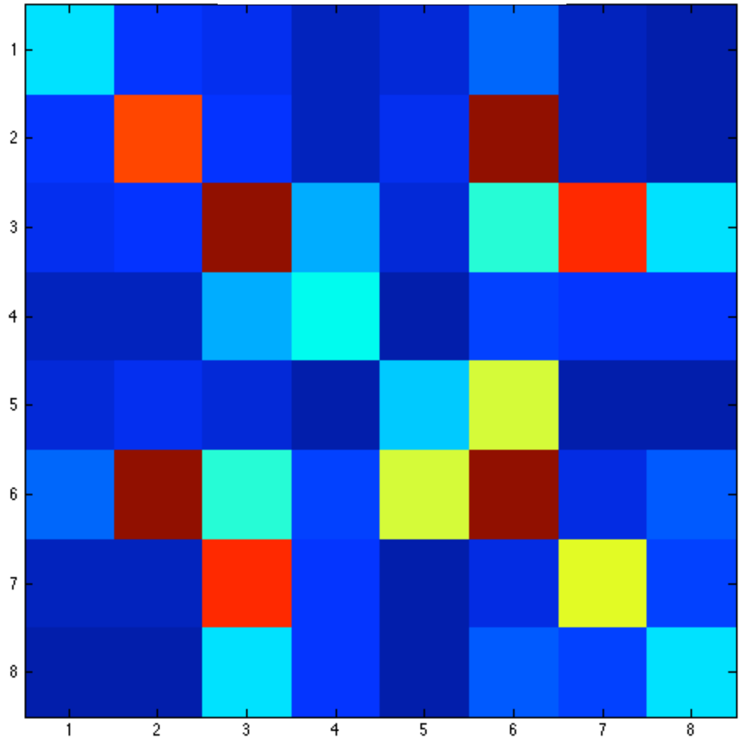
Noise covariance matrix

Example with test dataset

“Normal Coil”

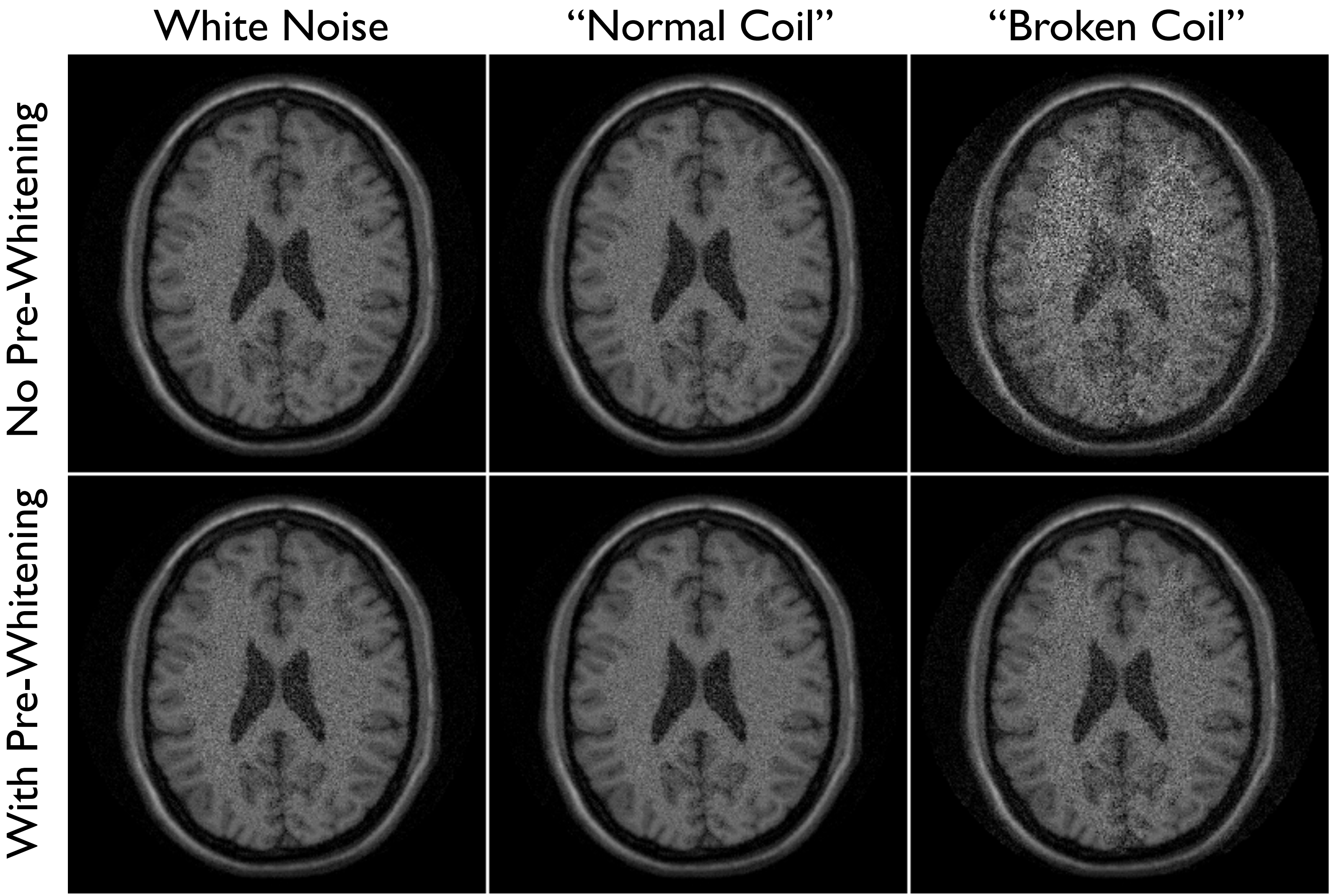


“Broken Coil”



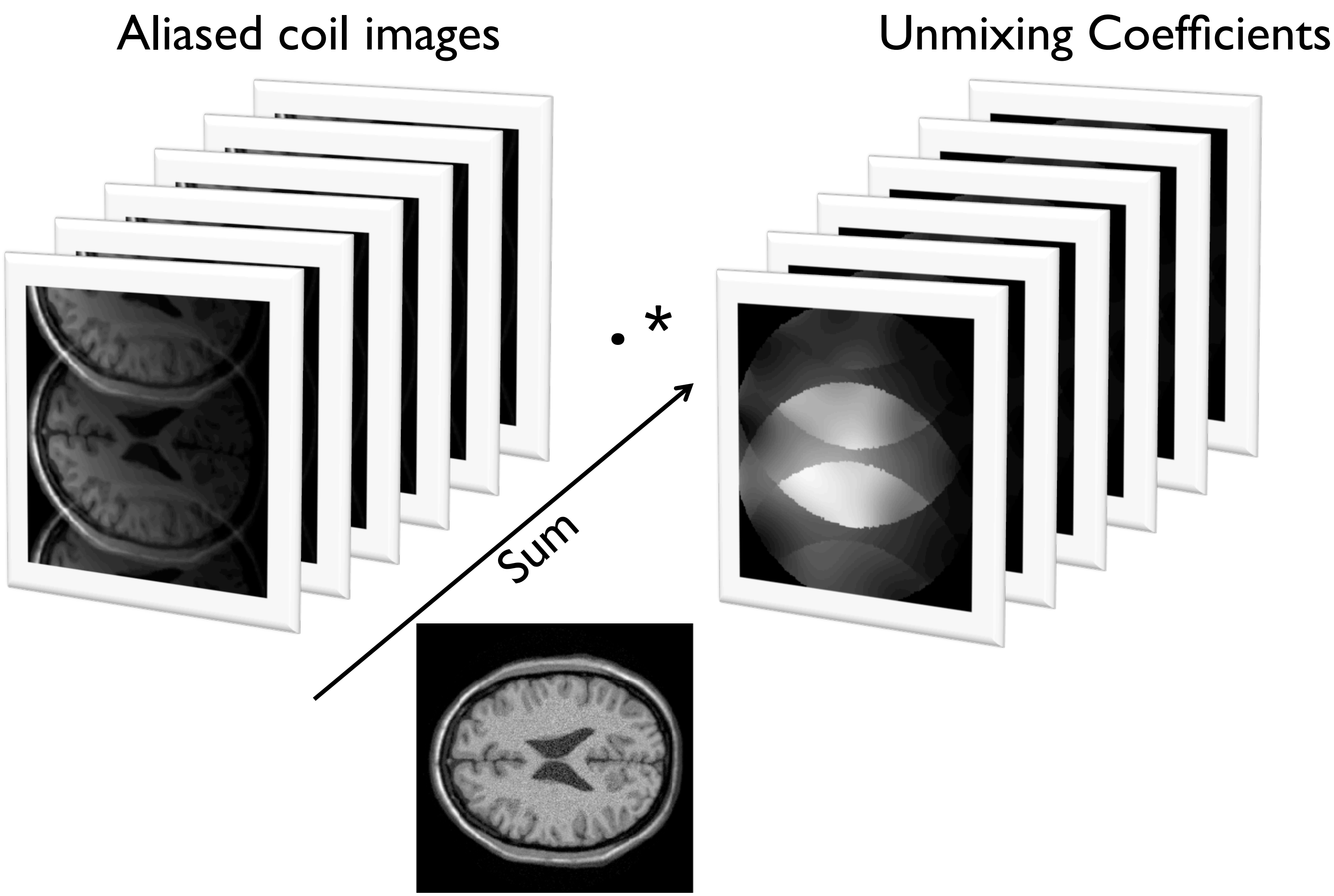
At least two broken pre-amps

Noise Pre-Whitening – SENSE Example



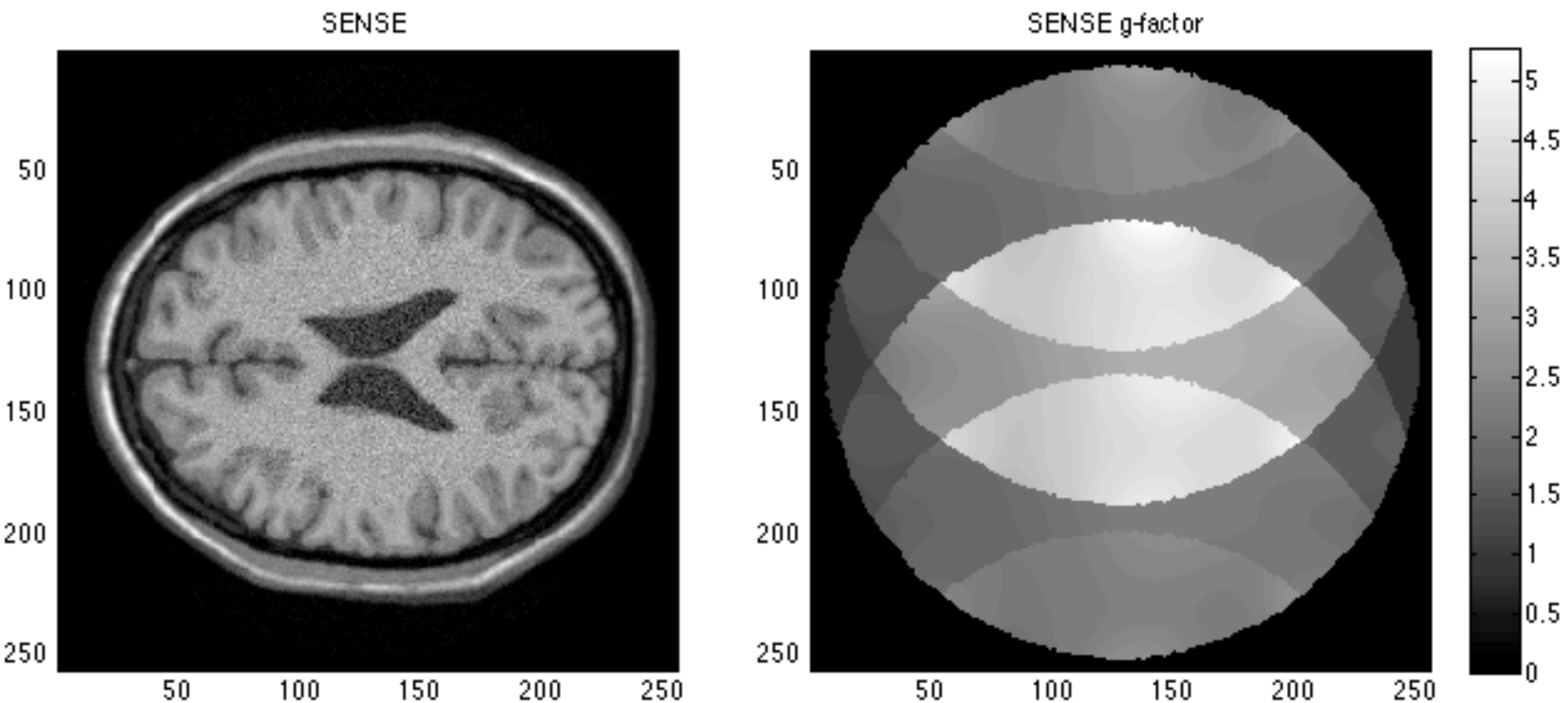
`ismrm_demo_noise_decorrelation.m`

SENSE – Image Synthesis with Unmixing Coefficients



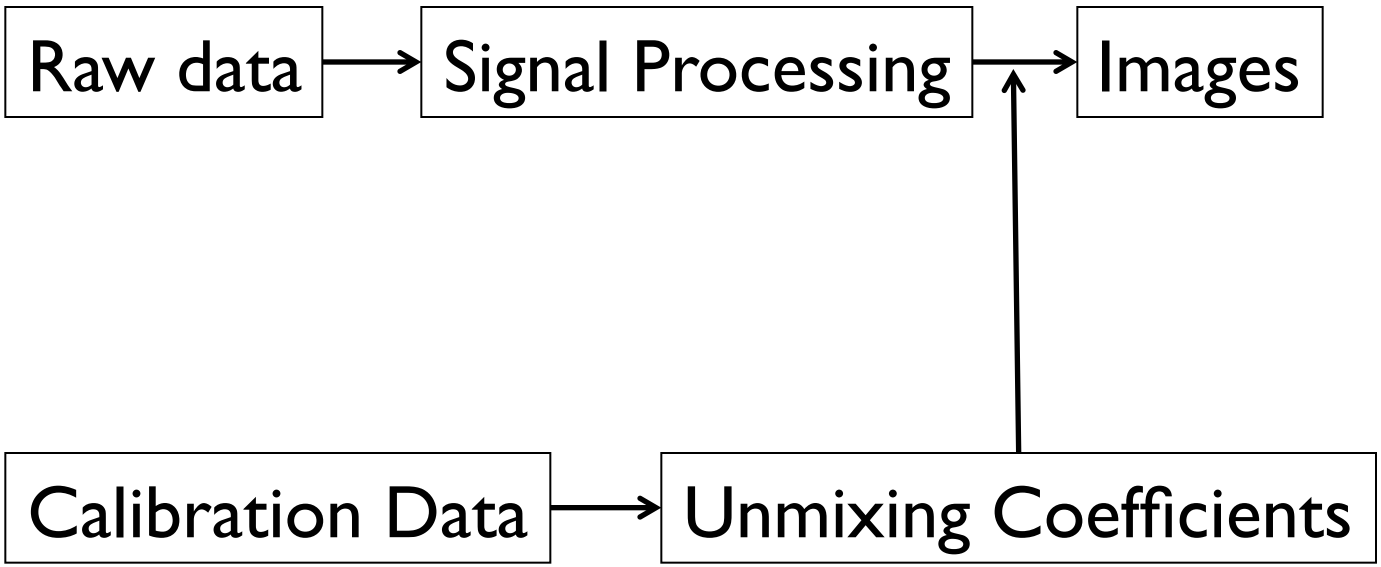
SENSE – Simple Rate 4 Example

$$\tilde{\rho}(x_1) = \sum_{i=0}^{N_c} u_i a_i \qquad g(x_1) = \sqrt{\sum_{i=0}^{N_c} |u_i|^2}\sqrt{\sum_{i=0}^{N_c} |S_i|^2}$$



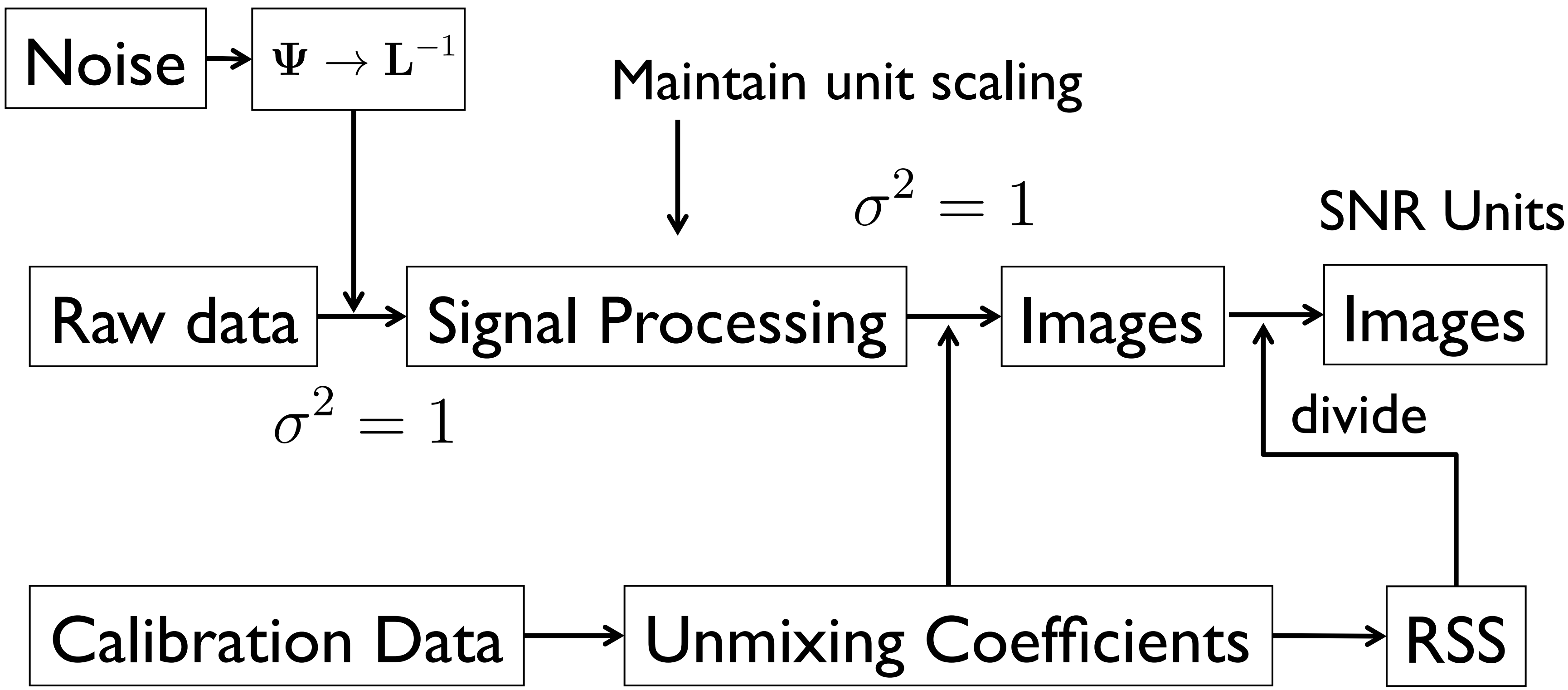
Reconstruction in SNR Units

Reconstruction Pipeline



Reconstruction in SNR Units

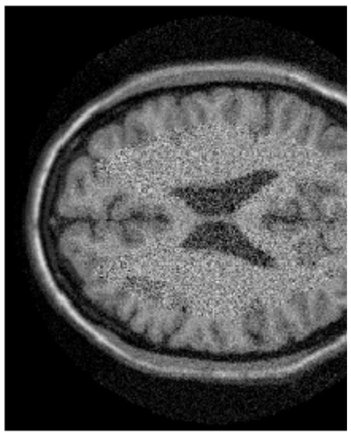
Reconstruction Pipeline

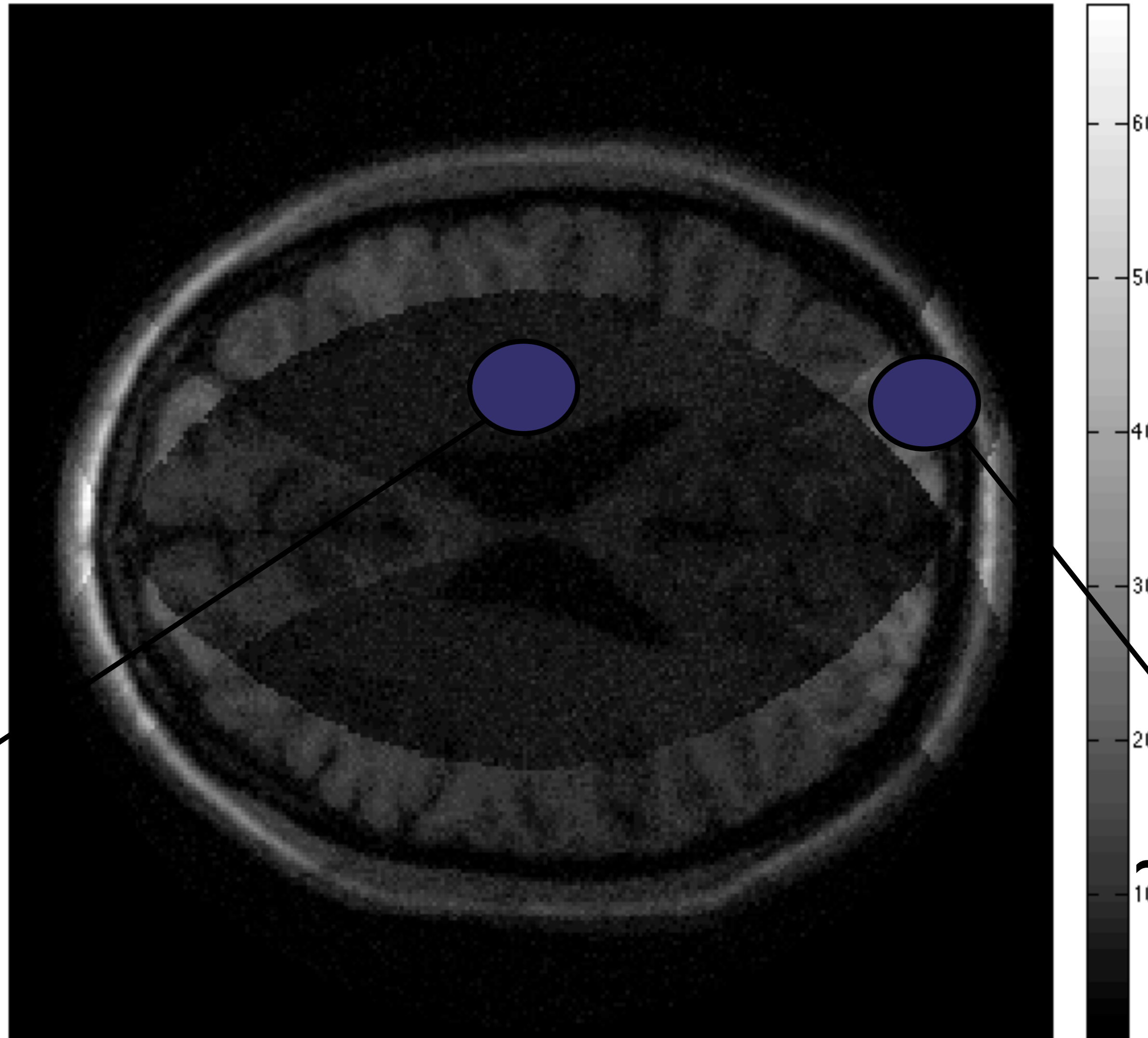


Kellman et al., Magnetic Resonance in Medicine 54:1439 –1447 (2005)

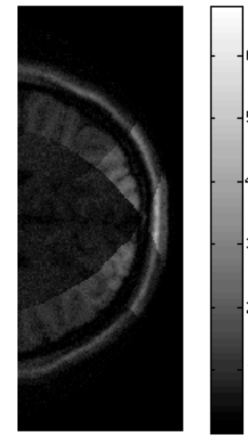
Reconstruction in SNR Units

Reconstruct





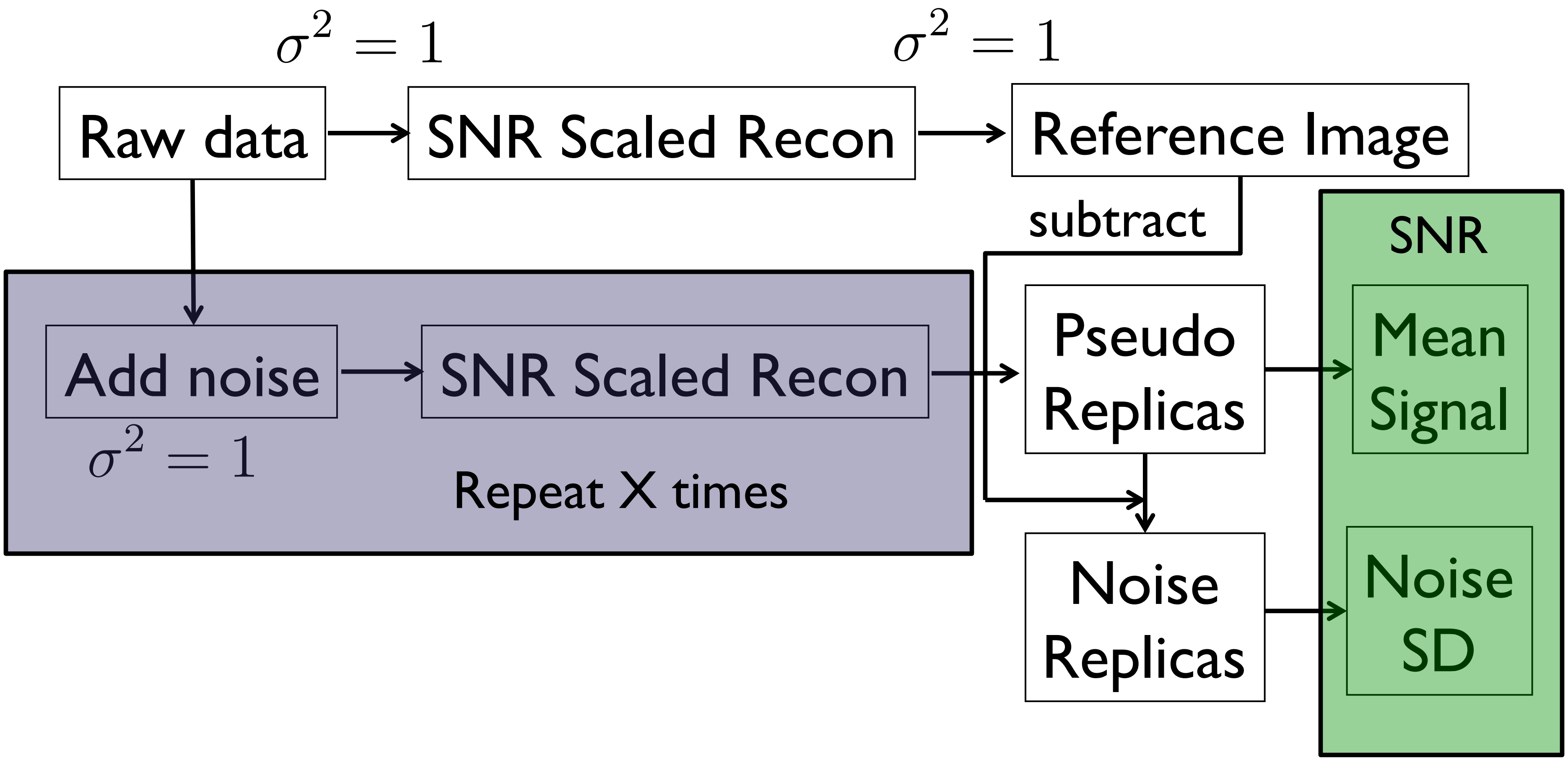
Jnits



–1447 (2005)

Pseudo-Replica Method

What if unmixing coefficients are never explicitly formed:

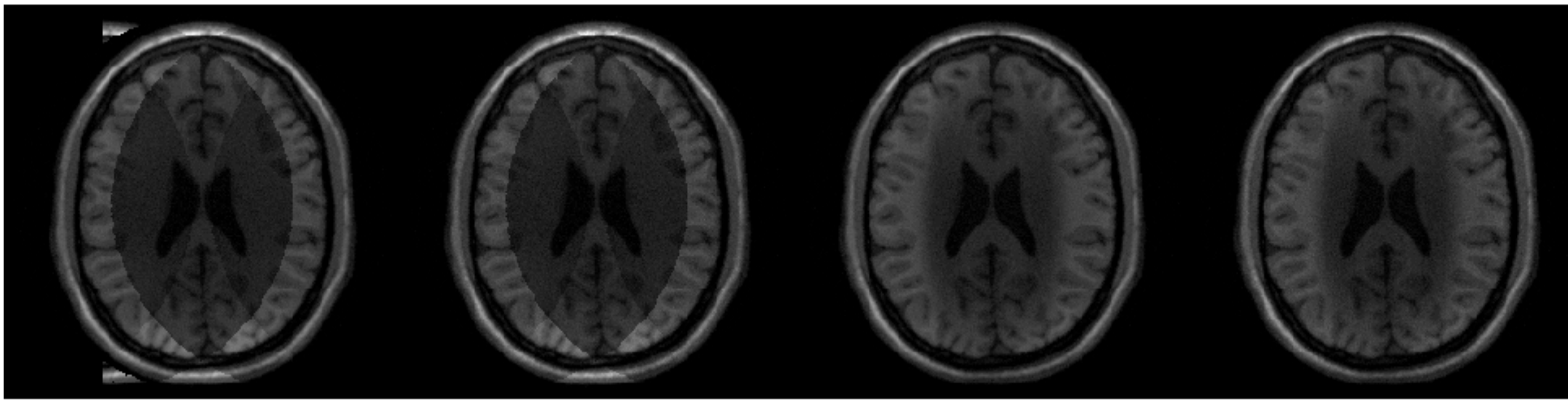


`ismrm_pseudo_replica.m`

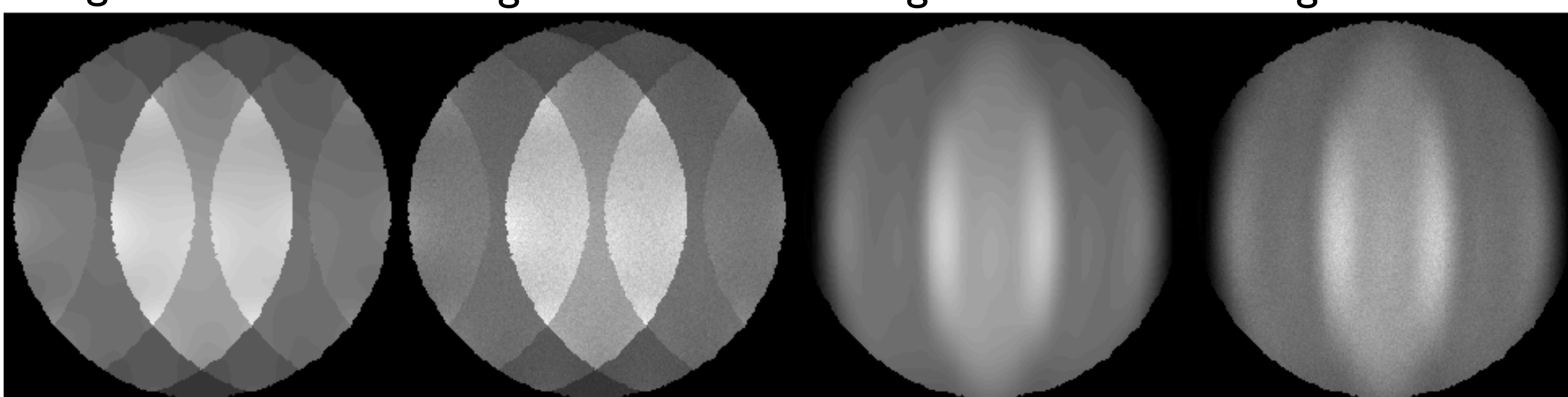
Pseudo-Replica Method – Example 256 trials

SENSE R4 GRAPPA R4

SNR UNMIX SNR PSEUDO SNR UNMIX SNR PSEUDO

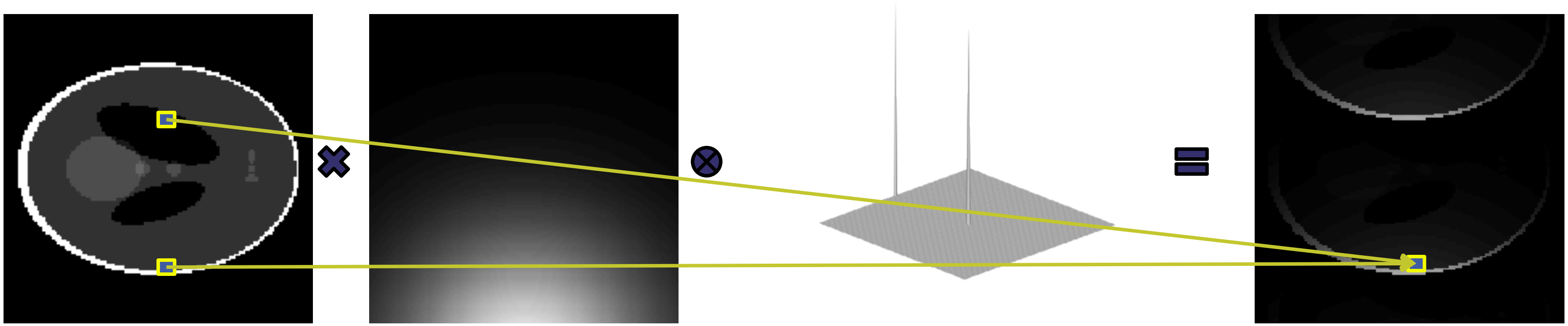


g UNMIX g PSEUDO g UNMIX g PSEUDO

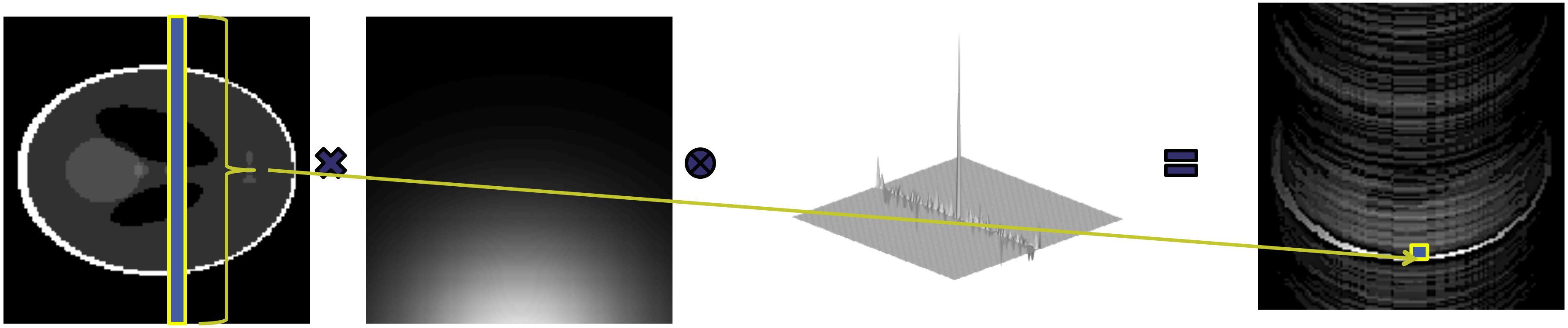


Advantage of Cartesian Undersampling

Cartesian Undersampling



“Random” Undersampling



Non-Cartesian Parallel MRI

To solve the general non-Cartesian case, we return to the original problem:

$$\mathbf{s} = \mathbf{E}\boldsymbol{\rho} \qquad\qquad \tilde{\boldsymbol{\rho}} = \arg\min_{\boldsymbol{\rho}} \left\{ \|\mathbf{E}\boldsymbol{\rho} - \mathbf{s}\|_2 \right\}$$

It is not practical to solve with direct inversion in general.

But we can use a number of different iterative solvers to arrive at the solution

- Conjugate Gradients
- LSQR (Matlab)

```
>> help lsqr
 lsqr   lsqr Method.
    X = lsqr(A,B) attempts to solve the system of linear equations A*X=B
    for X if A is consistent, otherwise it attempts to solve the least
    squares solution X that minimizes norm(B-A*X)...

    X = lsqr(AFUN,B) accepts a function handle AFUN instead of the matrix A.
    AFUN(X,'notransp') accepts a vector input X and returns the
    matrix-vector product A*X while AFUN(X,'transp') returns A'*X. In all
    of the following syntaxes, you can replace A by AFUN...
```

Iterative SENSE – First Cartesian

To use LSQR (or Conjugate Gradients), we "just" need to be able to write a function that does the multiplication with E and E^H:

Let's first look at a simple Cartesian case

Multiplication with E^H

```
rho = zeros(size(csm)); %csm: coil sensitivities
%sampling_mask: 1 where sampled, zero where not
rho(repmat(sampling_mask,[1 1 size(csm,3)]) == 1) = s(:);
rho = ismrm_transform_kspace_to_image(rho,[1,2]);
rho = sum(conj(csm) .* rho,3);
```

Multiplication with E

```
s = repmat(reshape(rho,size(csm,1),size(csm,2)),[1 1 size(csm,3)]) .* csm;
s = ismrm_transform_image_to_kspace(s, [1,2]);
s= s(repmat(sampling_mask,[1 1 size(csm,3)]) == 1);
```

Iterative SENSE

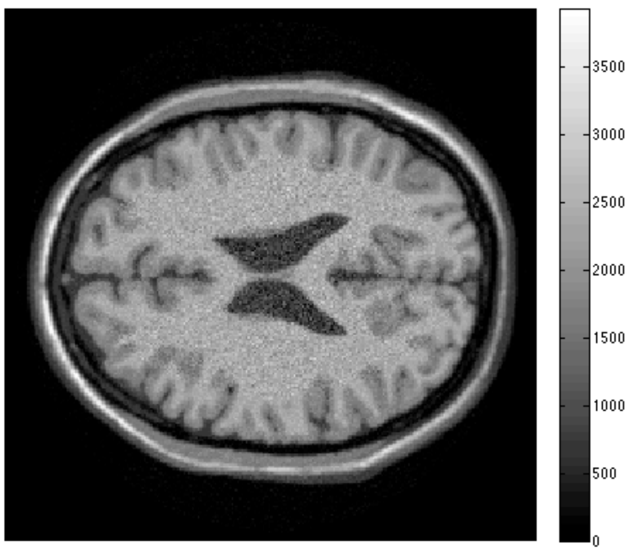
If we have the multiplication with E and E^H implemented as a Matlab function:

```
function o =  e_cartesian_SENSE(inp, csm, sp, transpose_indicator)
% sp: sampling pattern
% csm: coil sensitivities
```

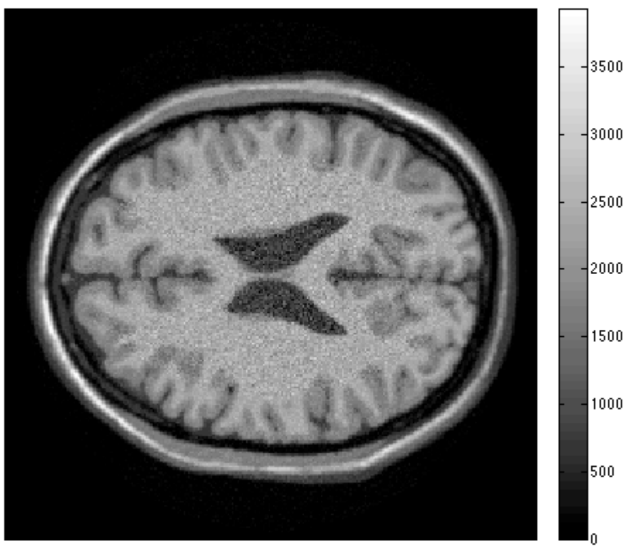
Iterative SENSE could be implemented as:

```
% s: vector of acquired data
E = @(x,tr) e_cartesian_SENSE(x,csm,(sp > 0),tr);
img = lsqr(E, s, 1e-5,50);
img = reshape(img,size(csm,1),size(csm,2));
```

Cartesian SENSE



Iterative SENSE



Quick note on the non-uniform FFT

To implement multiplication with E and E^H in the non-Cartesian case, we need to do the non-uniform Fourier transform[1,2].

In this course, we will use Jeff Fesslers "nufft" package. We recommend you download the latest version from:

http://web.eecs.umich.edu/~fessler/irt/fessler.tgz

```
%k: k-space coordinates [Nsamples, 2], range –pi:pi
%w: Density compensation weights
%s: Data

%Prepare NUFFT
N = [256 256]; %Matrix size
J = [5 5];     %Kernel size
K = N*2;       %Oversampled Matrix size
nufft_st = nufft_init(k,N,J,K,N/2,'minmax:kb');

recon = nufft_adj(s .* repmat(w,[1 size(s,2)]),nufft_st);
```

[1]Keiner, J., Kunis, S., and Potts, D. Using NFFT 3 - a software library for various nonequispaced fast Fourier transforms. ACM Trans. Math. Software, 2009
[2]Fessler J and Sutton B. Nonuniform fast Fourier transforms using min-max interpolation. IEEE TSP 2003

Iterative SENSE – non-Cartesian

To use LSQR (or Conjugate Gradients), we "just" need to be able to write a function that does the multiplication with E and E^H:

Now we have the tools for the non-Cartesian case:

Multiplication with E^H

```
samples = size(nufft_st.om,1); coils = numel(s)/samples;
s = reshape(s,samples,coils);
rho = nufft_adj(s .* repmat(sqrt(w),[1 coils]),nufft_st)./sqrt(prod(nufft_st.Kd));
rho = sum(conj(csm) .* rho,3);
rho = rho(:);
```

Ensure operators are adjoint

From `nufft_init`

Multiplication with E

```
s = repmat(reshape(rho,size(csm,1),size(csm,2)),[1 1 size(csm,3)]) .* csm;
s = nufft(s,nufft_st)./sqrt(prod(nufft_st.Kd));
s = s .*repmat(sqrt(w),[1 size(s,2)]);
s = s(:);
```

Iterative SENSE – non-Cartesian

If we have the multiplication with E and E^H implemented as a Matlab function:

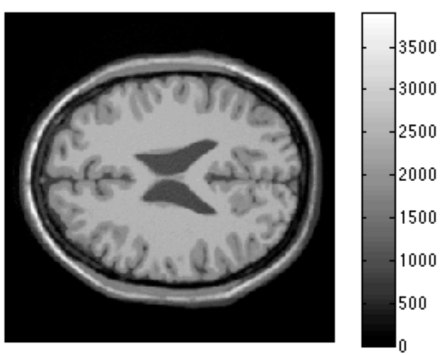
```
function o = e_non_cartesian_SENSE(inp, csm, nufft_st, w, transpose_indicator)
% nufft_st: From nufft_init
% csm: coil sensitivities, w: density compensation
```

Non-Cartesian SENSE could be implemented as:

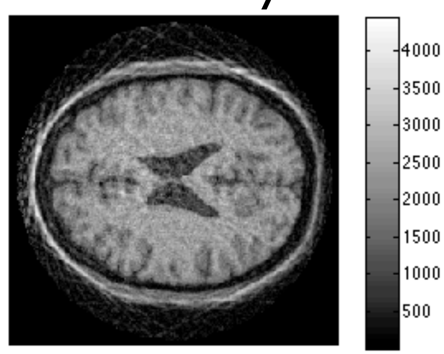
```
% s: vector of acquired data
E = @(x,tr) e_non_cartesian_SENSE(x, csm, nufft_st, w, tr);
img = lsqr(E, s .* repmat(sqrt(w),[size(csm,3),1]), 1e-3,30);
img = reshape(img,size(csm,1),size(csm,2));
```

Due to definition of E

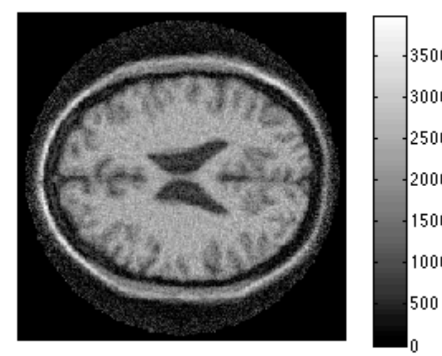
Fully sampled



24 projections
nufft only



24 projections
SENSE



Regularization - Basics

$$\mathbf{Ax} = \mathbf{b} \qquad \mathbf{x} = \left(\mathbf{A}^{\mathrm{H}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{H}}\mathbf{b} \qquad \mathbf{x} = \arg\min\left\{\|\mathbf{Ax}-\mathbf{b}\|_2\right\}$$

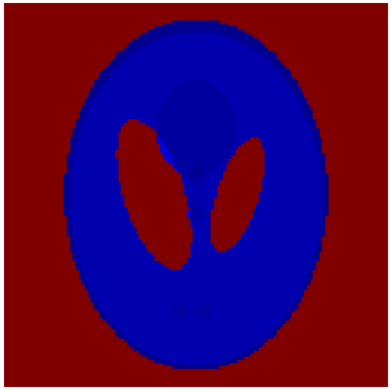
Adding constraints:

$$\mathbf{x}_\lambda = \arg\min\left\{ \|\mathbf{Ax}-\mathbf{b}\|_2 + \boxed{\lambda\|\mathbf{L}(\mathbf{x}-\mathbf{x}_0)\|_2}\right\}$$

$\mathbf{L}$: Linear Transform $\qquad$ $\mathbf{x}_0$: Prior Estimate

Solution:

$$\mathbf{x}_\lambda = \mathbf{x}_0 + \left(\mathbf{A}^{\mathrm{H}}\mathbf{A} + \lambda^2\mathbf{L}^{\mathrm{H}}\mathbf{L}\right)^{-1}\mathbf{A}^{\mathrm{H}}(\mathbf{b}-\mathbf{Ax}_0)$$

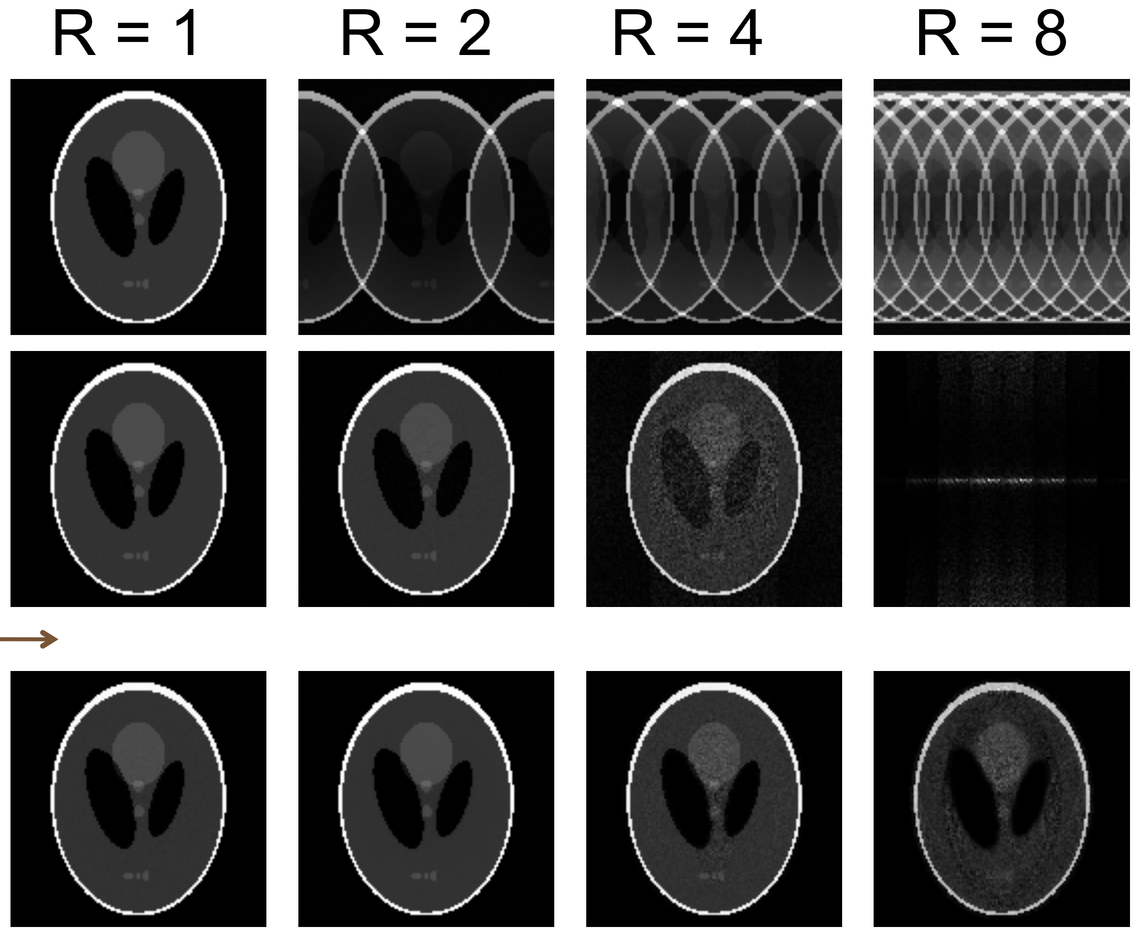


An example

$$\mathbf{x}_0 = 0 \qquad \mathbf{L} = \begin{bmatrix} 1/\rho_1 & & & \\ & 1/\rho_2 & & \\ & & \ddots & \\ & & & 1/\rho_N \end{bmatrix}$$

SENSE, 12 coils

$$\mathbf{L} = \begin{bmatrix} 1/\rho_1 & & & \\ & 1/\rho_2 & & \\ & & \ddots & \\ & & & 1/\rho_N \end{bmatrix}$$



Regularization – Iterative Methods

$$\tilde{\boldsymbol{\rho}} = \arg\min_{\boldsymbol{\rho}} \left\{ \|\mathbf{E}\boldsymbol{\rho} - \mathbf{s}\|_2 + \lambda \|\mathbf{L}\boldsymbol{\rho}\|_2 \right\}$$

Equivalent to solving:

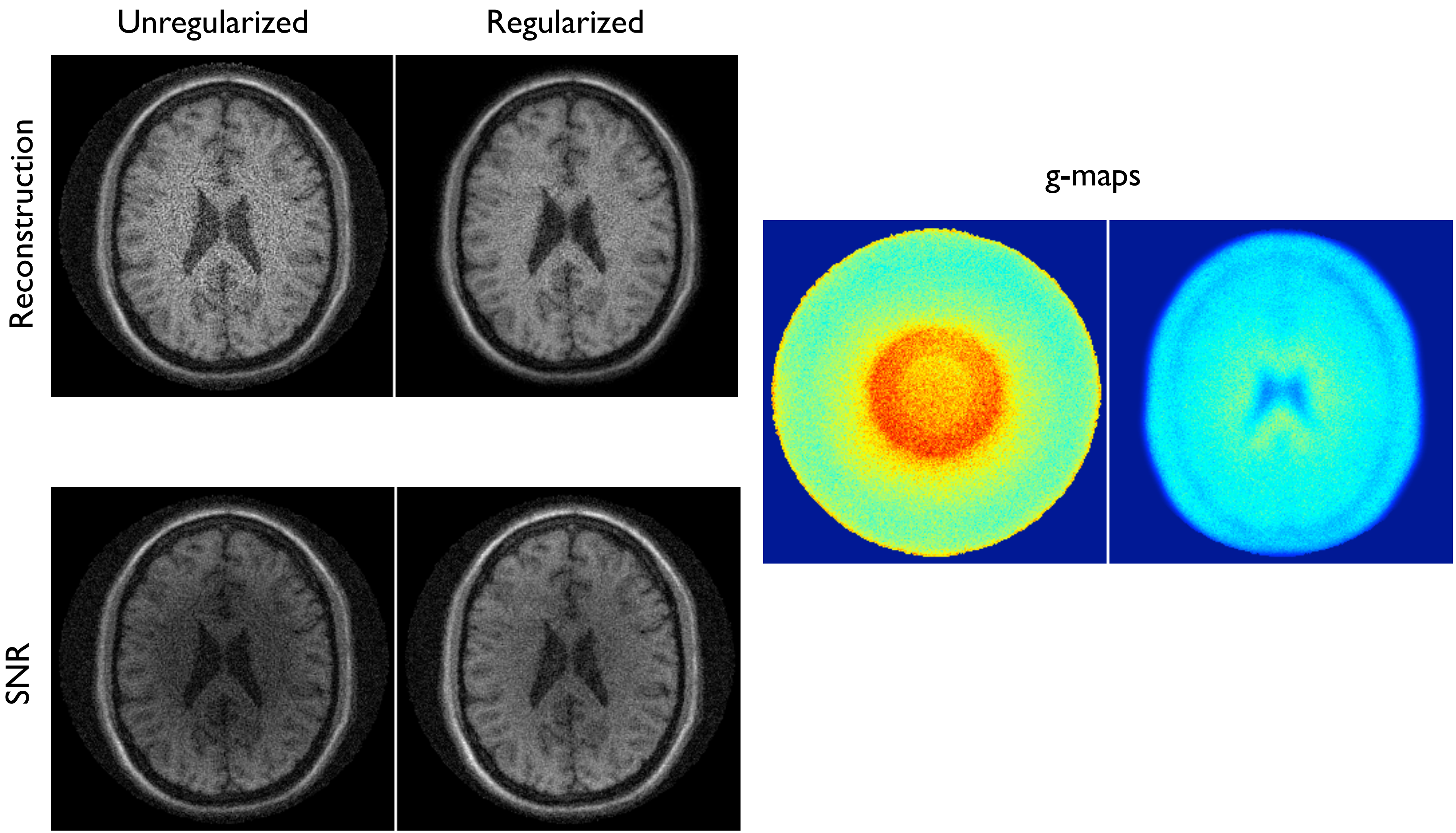
$$\begin{array}{r} \text{Measured data} \longrightarrow \\ \text{Vector of zeros} \longrightarrow \end{array} \begin{bmatrix} \mathbf{s} \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{E} \\ \mathbf{L} \end{bmatrix} \boldsymbol{\rho}$$

Regularized non-Cartesian SENSE could be implemented as:

```
% s: vector of acquired data
E = @(x,tr) e_reg_non_cartesian_SENSE(x, csm, nufft_st, w, tr);
img = lsqr(E, [s .* repmat(sqrt(w),[size(csm,3),1]);zeros(imgele,1)], 1e-3,30);
img = reshape(img,size(csm,1),size(csm,2));
```

`ismrm_demo_regularization_iterative_sense.m`

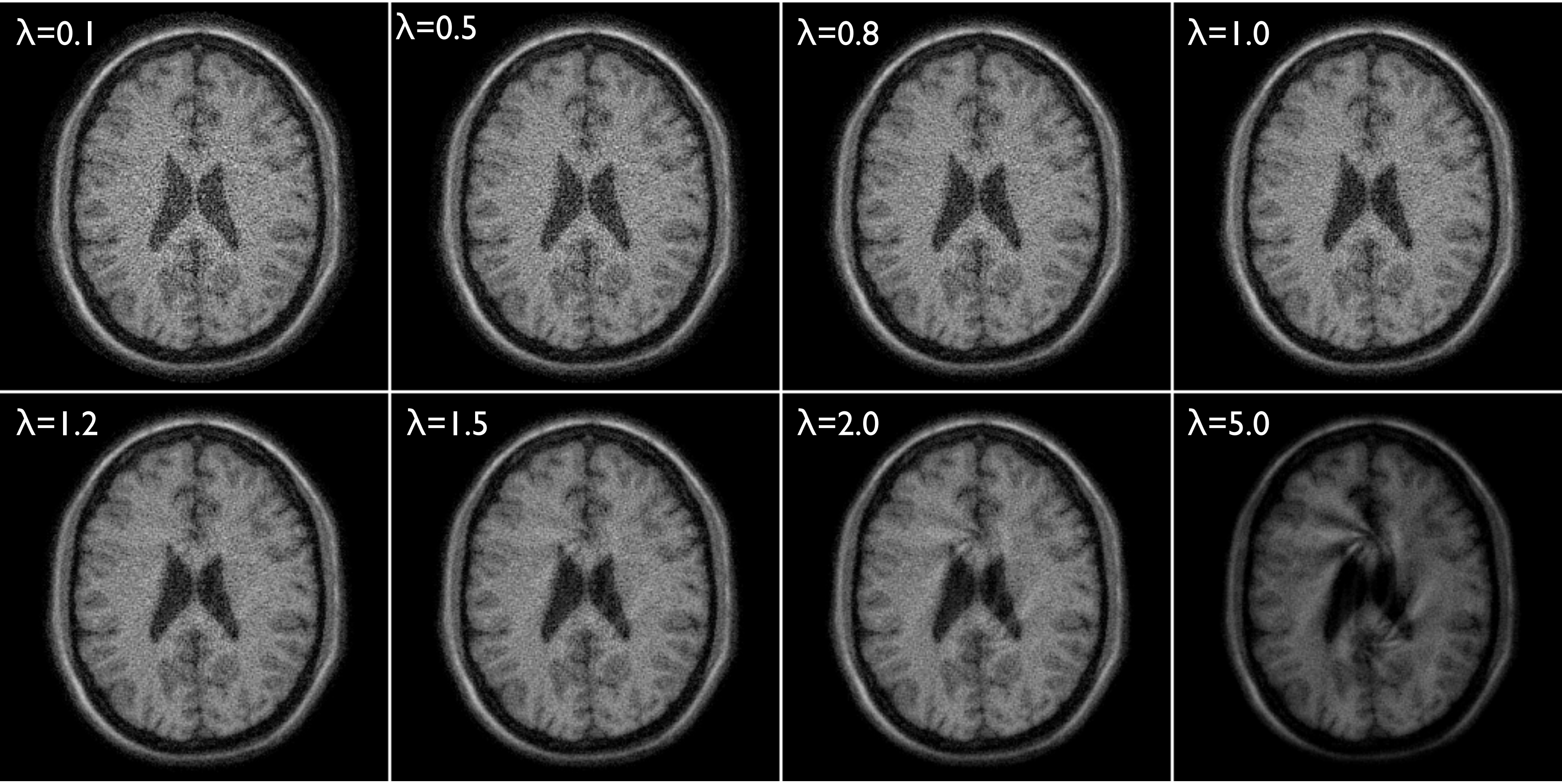
Regularization – Iterative Methods



`ismrm_demo_regularization_iterative_sense.m`

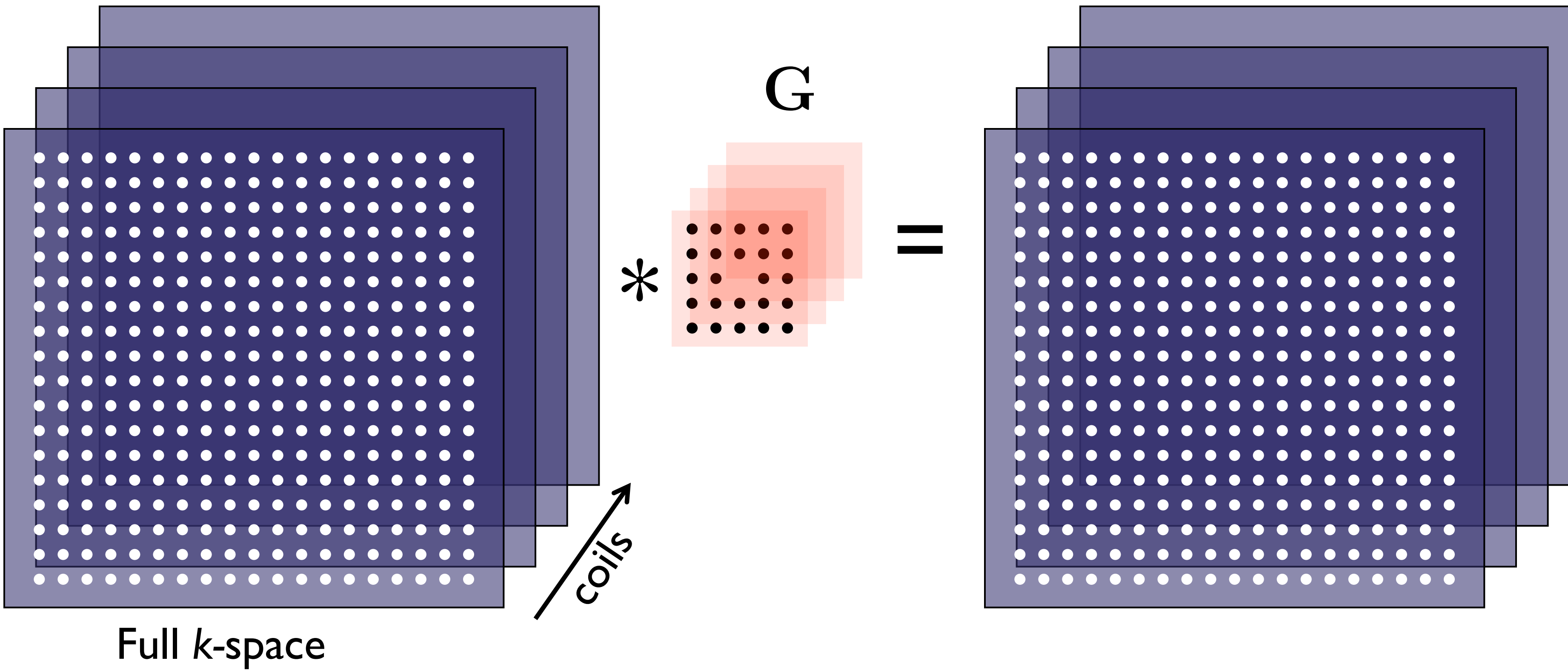
Regularization – Iterative Methods

$$\tilde{\boldsymbol{\rho}} = \arg\min_{\boldsymbol{\rho}} \{\|\mathbf{E}\boldsymbol{\rho} - \mathbf{s}\|_2 + \boxed{\lambda}\|\mathbf{L}\boldsymbol{\rho}\|_2\}$$



SPIRiT Approach

k-space points can be synthesized from neighbors



$$\mathbf{G}\mathbf{d} = \mathbf{d}$$

Lustig and Pauly. Magn Reson Med. 2010

SPIRiT Approach

We can formulate the reconstruction problem in *k*-space as:

$$\tilde{\mathbf{x}} = \arg\min_{\mathbf{x}} \{ \|\mathbf{D}\mathbf{x} - \mathbf{y}\|_2 + \lambda \|\mathbf{G}\mathbf{x} - \mathbf{x}\|_2 \}$$

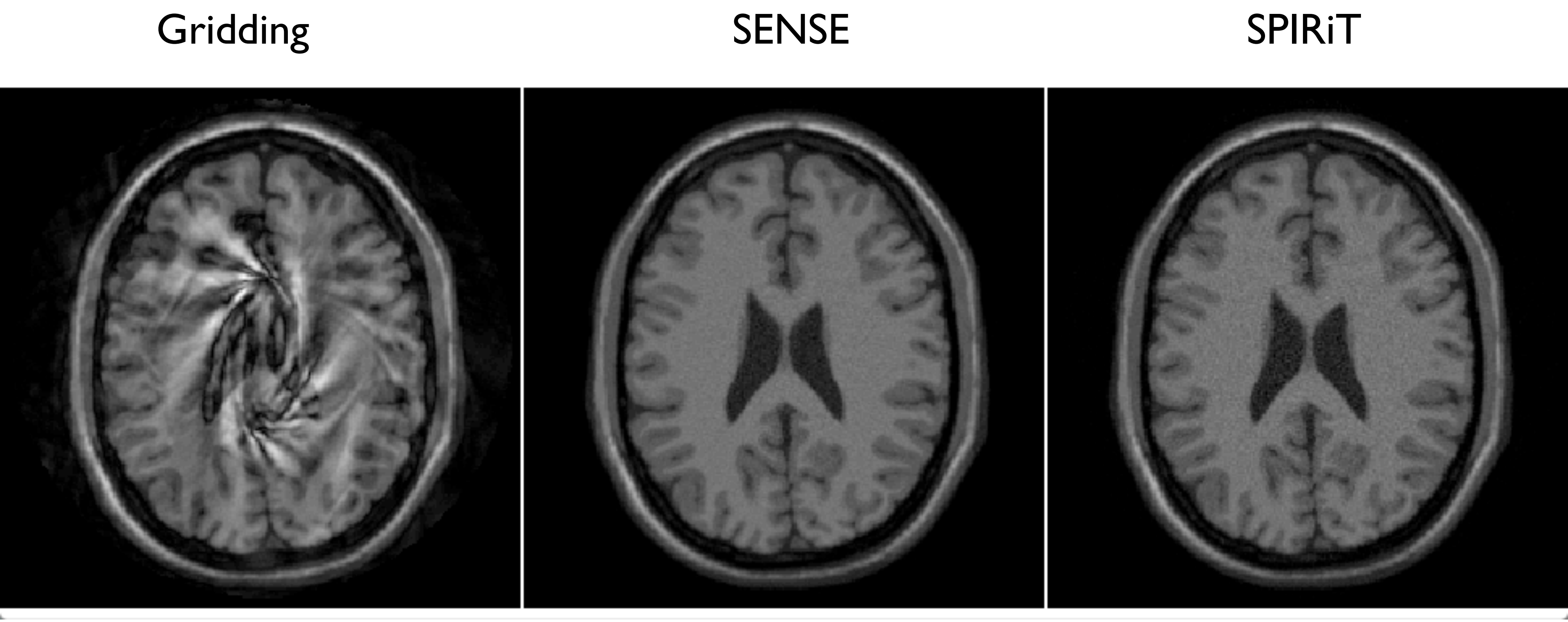
$\mathbf{x}$: Cartesian k-space solution.
$\mathbf{D}$: Sampling operator (e.g. onto non-Cartesian k-space)
$\mathbf{y}$: Sampled data
$\mathbf{G}$: SPIRiT convolution operator

Can be applied as multiplication in image space

Could also be sampling operator from image to *k*-space

Lustig and Pauly. Magn Reson Med. 2010

Spiral Imaging Example



`ismrm_demo_non_cartesian.m`

Summary

- Noise decorrelation is used to reduce the impact of varying noise levels in receive channels.
- SNR scaled reconstruction are a way to evaluate reconstructions directly on the images.
- Pseudo Replica Method allows the formation of SNR scaled images in methods where unmixing coefficients are not explicitly obtained
- Iterative methods can be used for both Cartesian and non-Cartesian methods
- Regularization can be added to iterative methods in a straightforward fashion

Acknowledgements

- Jeff Fessler
  - http://web.eecs.umich.edu/~fessler/code/
- Brian Hargreaves
  - http://mrsrl.stanford.edu/~brian/mritools.html
- Miki Lustig
  - http://www.eecs.berkeley.edu/~mlustig/Software.html

Download code, examples:

http://gadgetron.sf.net/sunrise

EXERCISES

1. Getting Started

Load exercise data

```
load hansen_exercises.mat
whos
```

Reconstruct aliased images

- Observations, noise?

Do SENSE reconstruction

- Calculate SENSE unmixing
- Apply unmixing

2. Noise

Generate noise covariance matrix

- noise_color
- Observations, is this a good coil?

Do noise pre-whitening

```
help ismrm_calculate_noise_decorrelation_mtx
help ismrm_apply_noise_decorrelation_mtx
```

Do SENSE reconstruction

- Compare to before prewhitening

3. SNR Scaled Reconstruction

Analyse FFT to image space.

- Scaling?
- How to set the scale factor

Do SENSE reconstruction

Create SNR image and g-map

4. Pseudo replica method

Do 100 reps of SENSE recon (just unmixing part)

Calculate standard deviation of the noise

Create SNR image and g-map

5. Non-Cartesian

Reconstruct aliased images using nufft

Setup encoding matrix anonymous function

```
ismrm_encoding_non_cartesian_SENSE.m
```

Reconstruct non-Cartesian SENSE

Explore non-Cartesian Demo

```
ismrm_demo_non_cartesian.m
```